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On a Class of Distance Transitive Graphs

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A class of distance transitive graphs characterized in terms of graphs of projective spaces. It is shown that if $p \neq 5$ is a prime and $\mathscr{G}$ is a distance transitive bipartite graph of diameter 3 and valence p, then $\mathscr{G}$ is the bipartite graph of a suitable projective space design.

1. INTRODUCTION

In recent times, considerable attention has been given to the study of distance-transitive graphs [1], but a satisfactory classification seems still too far away. In what follows, a class of distance-transitive graphs is characterized in terms of graphs of projective spaces.

2. PRELIMINARIES

Let $\mathscr{G}$ be a simple connected graph of diameter ν on the vertex set V of v elements. Let $\delta: V \times V \to \mathbb{N} \cup \{0\}$ be the usual distance function on V. For $x, y \in V$, $\delta(x, y)$ is the length of the shortest path in $\mathscr{G}$ between x and y. Let G denote the group of automorphisms of $\mathscr{G}$.

2.1. *Definition*

$\mathscr{G}$ is said to be distance transitive if and only if, for each pair (x, y), $(a, b) \in V \times V$ with $\delta(x, y) = \delta(a, b)$, there exists a $g \in G$ such that

$$g(x) = a \quad \text{and} \quad g(y) = b.$$

2.2. *Examples*

Several examples of such graphs with small diameters are known. See [1, 5]. We shall discuss two examples of such graphs in detail.

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EXAMPLE 1. Consider the projective space $PG(d, q)$ of dimension d over the finite field of q elements. Let $\mathbb{P}$ denote the set of points and $\mathbb{B}$ the set of hyperplanes of this space. It is well known that $(\mathbb{P}, \mathbb{B})$ define a projective design (also called symmetric balanced incomplete block design) [3], with point set $\mathbb{P}$ and block set $\mathbb{B}$. Let $\mathscr{G}_{d,q}$ be the graph, whose vertex set consists of $\mathbb{P} \cup \mathbb{B}$ and in which two vertices are adjacent if and only if one of them is in $\mathbb{P}$, the other is in $\mathbb{B}$, and they are incident. This graph has diameter 3, is bipartite and is distance transitive.

EXAMPLE 2. Consider the finite field F with $q = 11$ elements. The set of squares on F is $B_1 = \{1, 3, 4, 5, 9\}$. For $0 \leqslant i \leqslant 11$, let $B_i = \{1 + i, 3 + i, 4 + i, 5 + i, 9 + i\}$ and let $\mathbb{B} = \{B_1, B_2, ..., B_{11}\}$. Then $\mathbb{B}$ is the set of blocks of a projective design—in particular, a Hadamard design—on the point set F. As in Example 1, one may construct a graph, denoted by $\mathscr{G}_{11}$, whose vertex set is $F \cup \mathbb{B}$ and in which two vertices are adjacent if and only if one of them is in F, the other is in $\mathbb{B}$, and they are incident. This graph also has diameter 3 and is distance transitive.

It is conceivable that these two types of examples will exhaust all distance transitive bipartite graphs of diameter 3. We shall show that this is in fact the case, when the graph has a prime valence.

3. POLYNOMIAL OF A GRAPH

Let $\mathscr{G}$ be a distance-transitive bipartite graph of diameter 3. Let A be the 0–1 matrix of size $v \times v$, where $A(i,j) = 1$ whenever the ith and jth vertices of $\mathscr{G}$ are adjacent and $A(i,j) = 0$ otherwise. A is called the adjacency matrix of $\mathscr{G}$. Let x, y be two vertices of $\mathscr{G}$ and let $p_{jk}(x, y)$ be the number of vertices in V that are at distance j from x as well as at distance k from y, $0 \leqslant j, k \leqslant 3$.

LEMMA 3.1. *The number $p_{jk}(x, y)$ is independent of x and y, but depends only on the distance between x and y.*

Proof. Let $x, y; a, b$ be vertices of R such that $\delta(x, y) = \delta(a, b) = i$, $0 \leqslant i \leqslant 3$. Let $S(j, k)$ be the set of vertices at distance j from x and k from y. By distance transitivity, there exists an automorphism g of $\mathscr{G}$ such that $(g(x), g(y)) = (a, b)$. Since g preserves distances, $gS(j, k)$ is a subset of the set of vertices at distance j from a and k from b. Hence $p_{jk}(x, y) \leqslant p_{jk}(a, b)$. By reversing the argument, we have $p_{jk}(a, b) \leqslant p_{jk}(x, y)$ and the result follows. Following the notation in [2], when x and y are two vertices at distance i then $p_{jk}(x, y)$ shall be denoted by p^i_{jk}, $0 \leqslant i, j, k \leqslant 3$, and p^0_{ii} shall be denoted by n_i.

LEMMA 3.2. *The following relations hold for the parameters* p_{jk}^i.

(i) $p_{jk}^i = 0$ *whenever* $j + k < i$, $0 \leqslant i, j, k \leqslant 3$.

(ii) $p_{jk}^0 = 0$ *for* $j \neq k$, *and* $n_i p_{jk}^i = n_j p_{ik}^j = n_a p_{ij}^k$, $0 \leqslant i, j, k \leqslant 3$.

(iii) $p_{11}^1 = p_{21}^2 = 0$.

Proof. For (i), consider two vertices x and y at distance i. If $p_{jk}^i > 0$, then there exists a vertex z at distance j from x and k from y. This gives a path of length $j + k$ from x to y and by definition of distance, one must have $j + k - i$.

To prove (ii), observe that there are n_i vertices at distance i from a given vertex and that $vn_i p_{jk}^i$ counts the number of triples of the form (x, y, z) where $\delta(x, y) = i$, $\delta(y, z) = k$, $\delta(x, z) = j$.

Now, a bipartite graph contains no odd cycles and if either p_{11}^1 or p_{21}^2 were not zero, then there would be a cycle of length 3 or 5. Hence (iii).

Let J denote the $v \times v$ matrix, each of whose entry is 1. We shall express J as a sum of multiples of powers of A and thereby compute the minimum polynomial of A. For this, define a 0–1 matrix A_t, $0 \leqslant t \leqslant 3$, where $A_t(i,j) = 1$ whenever the ith and jth vertices of $\mathscr{G}$ are at distance t and $A_t(i,j) = 0$ otherwise. Here A_1 is A and A_0 is the identity matrix.

LEMMA 3.3. *The matrices* A_t, $0 \leqslant t \leqslant 3$, *satisfy the following equations*

$$\sum_{t=0}^{3} A_t = J, \tag{1}$$

$$A_j A_k = \sum_{i=0}^{3} p_{jk}^i A_i. \tag{2}$$

Proof. The first equation is obvious and as to the second, consider

$$A_j A_k(l, m) = \sum_t A_j(l, t)\, A_k(t, m).$$

The right side counts the number of vertices at distance j from the lth vertex and at distance k from the mth vertex. If the lth and mth vertices are at distance i then this number is p_{jk}^i. This verifies the second equation.

Using these two equations along with Lemma 3.2, we have $A_1^2 = n_1 A_0 + p_{11}^2 A_2$. Multiplying by A_1 and using Eq. (2),

$$A_1^3 = n_1 A_1 + p_{11}^2 [p_{12}^1 A_1 + p_{12}^3 A_3].$$

Using the last two equations and substituting for A_2 and A_3 in Eq. (1) we get

$$\phi(A_1) = A_1^3 + p_{12}^3 A_1^2 - [n_1 - p_{11}^2(n_1 - 1 - p_{12}^3) A_1] + p_{12}^3(p_{11}^2 - n_1) A_0$$
$$= p_{11}^2 p_{12}^3 J. \quad (3)$$

$\phi(x)$ is the Hoffman polynomial of the graph $\mathscr{G}$ [4].

Since $A_1 = A$ and J are commuting symmetric matrices, they can be simultaneously diagonalized through conjugation by a suitable nonsingular matrix. Since v is a simple eigenvalue of J and the only other eigenvalue of J is 0, it can be seen that the diagonal entry in the reduced from of A, corresponding to v in the reduced form of J is n_1. Hence from (3), we have

$$n_1^3 + p_{12}^3 n_1^2 - [n_1 - p_{11}^2(n_1 - 1 - p_{12}^3)]\, n_1 + p_{12}^3(p_{11}^2 - n_1) = P_{11}^2 p_{12}^3 v. \quad (4)$$

Also, since $(A - n_1 I)J = 0$, $(x - n_1)\,\phi(x)$ is the minimal polynomial of A. Since R is bipartite, $-n_1$ is also an eigenvalue of $\mathscr{G}$ and so $\phi(-n_1) = 0$. Or,

$$-n_1^3 + p_{12}^3 n_1^2 + [n_1 - p_{11}^2(n_1 - 1 - p_{12}^3)]\, n_1 + p_{12}^3(p_{11}^2 - n_1) = 0. \quad (5)$$

From (4) and (5)

$$\frac{v}{2} = \frac{n_1(n_1 - 1)}{p_{11}^2} + 1. \quad (6)$$

Since $\mathscr{G}$ is bipartite, consider a partition of $V = V_1 \cup V_2$ where, no two vertices in V_i are adjacent, $i = 1, 2$. Since R is regular of valence n_1, $| V_1 | = | V_2 | = v/2$. Also the number of vertices at distance 2 from a given vertex is $n_1(n_1 - 1)/p_{11}$. Hence (6) shows that any two vertices lying in V_1 (or V_2) are at distance 2 from each other. Consider the incidence structure $\pi = (V_1, V_2, I)$ where

$$I: V_1 \times V_2 \to \{0, 1\} \text{ such that for } (x, y) \in V_1 \times V_2,$$
$$\begin{aligned} I(x, y) &= 1, &&\text{if } \delta(x, y) = 1, \\ &= 0, &&\text{otherwise.} \end{aligned}$$

From the discussion above, it is clear that the incidence structure π defines a projective design with point set V_1; block set V_2 each block having $k = n_1$ points and any two points being incident with $\lambda = p_{11}^2$ blocks.

4. Main Theorem

LEMMA 4.1. *The design π has a collineation group which is doubly transitive on the points (and blocks).*

Proof. Let g be an automorphism of the graph R such that for some $x \in V_1$, $g(x) \in V_1$. Let y be any other element of V_1. It was shown that V_1 consists only of vertices at distance 2 from a given vertex in V_1 and since g preserves distances, $\delta(g(x), g(y)) = \delta(x, y) = 2$. But $g(x) \in V_1$ and so $g(y) \in V_1$. In other words, g is a mapping of V_1 to V_1 and obviously preserves incidence in π. Hence g is a collineation of π. For x, y; $a, b \in V_1$, $x \neq y$, $a \neq b$, there exists an automorphism h of $\mathscr{G}$ such that $(h(x), h(y)) = (a, b)$. From what is said earlier, h is a collineation of π and so the subgroup of the automorphism group of $\mathscr{G}$, fixing V_1 setwise is a doubly transitive collineation group of π. The same argument shows that this group is doubly transitive on blocks of π.

It is known that [3, Sect. 35] a projective design with k points in a block and λ blocks through 2 points, with a doubly transitive collineation group is one of the designs discussed in Examples 1 or 2 of Section 2.2, whenever either k or $k - \lambda$ is a prime. Using this result with Lemma 4.1, we have the following theorem.

THEOREM. *If $\mathscr{G}$ is a distance transitive bipartite graph of diameter 3 and valence n, then* R *is isomorphic to either $\mathscr{G}_{d,q}$ for some d and q, or $\mathscr{G}_{11}$, provided that either n or $n - p_{11}^2$ is a prime.*

Remark. Since $\mathscr{G}_{11}$ is not isomorphic to $\mathscr{G}_{d,q}$ for any d or q, it follows that $\mathscr{G}_{11}$ is the unique distance transitive bipartite graph of valence 5 and diameter 3.

REFERENCES

1. N. BIGGS, "Algebraic Graph Theory," Cambridge tracts in Math. 67, Oxford Univ. Press, London/New York, 1974.
2. R. C. BOSE, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.* **13**, 1963.
3. P. DEMBOWSKI, "Finite Geometries," Springer-Verlag, Berlin/New York, 1968.
4. A. J. HOFFMAN, On the polynomial of a graph, *Amer. Math. Monthly* **70** (1963), 30–36.
5. W. T. TUTTE, A family of cubical graphs. *Proc. Cambridge Phil. Soc.* **43**, 459–474.